Master on Physics and Engineering on Fusion Plasmas
Introductory Atomic and Molecular Physics. Problems (3).

1. Show that the fundamental equation of the classical mechanics:
$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$$
is satisfied by the expectation values of the corresponding quantum mechanical operators.

2. Show that the momentum eigenfunctions for a one-dimensional free particle are:
$$u_p = h^{-1/2} \exp\left(\mathrm{i}px/\hbar\right)$$
Hint: use the definition of the Dirac delta function:
$$\delta(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x e^{\mathrm{i}\sigma x}$$
and assume that the continuum wavefunctions are normalized as:
$$\int u^*_{f'} u_f \mathrm{d}^3\boldsymbol{r} = \delta(f' - f)$$

3. Let $< A >$ be the expectation value of a time-independent operator. Using the time-dependent Schrödinger equation, write down the equation for $\frac{\mathrm{d}<A>}{\mathrm{d}t}$. Consider the particular cases of $< x_k >$ and $< p_k >$.

4. Let A, B and H be operators on E_2 and $\{|1>, |2>\}$ an orthonormal basis set of this space. One has:
$$A|1>= m|1>;\ B|1>= |2>;\ H|1>= \mathrm{i}E|2>$$
$$A|2>= -m|2>;\ B|2>= |1>;\ H|2>= -\mathrm{i}E|1>$$
   (a) Write down these operators in terms of bras and kets.
   (b) Find the conmutation rules of these operators.
   (c) Find the matrix representation of these operators in the basis set of eigenestates of the operator B.

5. Let $\{|u_i>\}$ (i=1,2,3) be an orthonormal basis of $\mathbb{R}^3$. The matrices associated to the operators H and B are:
$$\mathbf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathbf{B} = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
Show that $[H, B] = 0$ and find a basis of simultaneous eigenvectors of H and B. Evaluate $[H, B^2]$ and $[H^2, B]$. Are $\{ H\}$, $\{ B\}$, $\{ H, B\}$, $\{ H^2, B\}$ complete sets of commuting observables?

6. Consider the set of orthonormal vectors $\{|u_i >\}$ (i=1,2,3) and the operators L_z and S defined through:

$$L_z|u_1 >= |u_1 >;\ L_z|u_2 >= 0;\ L_z|u_3 >= -|u_3 >$$
$$S|u_1 >= |u_3 >;\ S|u_2 >= |u_2 >;\ S|u_3 >= |u_1 >$$

Write down the matrices for the operators L_z, L_z^2, S and S^2 in this basis. Are $\{L_z^2, S\}$ a complete set of commuting observables?

7. Consider the ladder operators $L_\pm = L_x \pm \mathrm{i}L_y$. Show that:

   (a) $[L_z, L_+] = L_+;\ [L_z, L_-] = -L_-$.

   (b) $L^2 = L_-L_+ + L_z + L_z^2 = L_+L_- - L_z + L_z^2$.

8. Consider the set $\{u_{jm}\}$ of the simultaneous eigenfunctions of L_z and L^2. Prove that $L_\pm u_{jm}$ are also eigenfunctions of L_z and L^2.

9. The ladder operators $J_\pm$ fulfill:

$$J_\pm u_{jm} = c_\pm u_{jm\pm1}$$

Find the coefficients $c_\pm$ which lead to normalized functions $u_{jm\pm1}$.

10. Write down the matrix representations of the ladder operators $J_\pm$ in the basis set of simultaneous eigenfunctions of J^2 and J_z with $j = 1/2$.